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Quantities which define conically self-similar free-vortex solutions to the Navier–Stokes equations uniquely

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It is proved that if, in addition to the opening angle of the bounding conical streamsurface and the circulation thereon, one of the radial velocity, the radial tangential stress or the pressure on the bounding streamsurface is given, then a conically self-similar free-vortex solution is uniquely determined in the entire conical domain. In addition, it is shown that for flows inside a cone the same conclusion holds for the Yih *et al.* (1982) parameter T, but for exterior flows it is shown numerically that non-uniqueness may occur. For given values of the opening angle of the bounding conical streamsurface and the circulation thereon the asymptotic analysis of Shtern & Hussain (1996) is applied to obtain asymptotic formulae which interrelate the opening angle of the cone along which the jet fans out and the radial tangential stress on the bounding surface. A striking property of these formulae is that the opening angle of the cone along which the jet fans out is independent of the value of the viscosity as long as it is small enough for the first-order asymptotic expressions to apply. However, these formulae are shown to be inaccurate for moderate values of the ratio of the circulation at the bounding surface and the viscosity. To amend this shortcoming, an alternative, more accurate, asymptotic analysis is developed to derive second-order correction terms, which considerably improve the accuracy.

1. Introduction

Within the class of conically self-similar solutions to the Navier–Stokes equations, which was originally discovered by Long (1958, 1961) and independently by Goldshtik (1960), some features of swirling flows can be studied in a greatly simplified framework. There are essentially two kinds of conically self-similar solutions to the Navier–Stokes equations: forced vortex solutions, which satisfy the no-slip condition at some conical boundary $\theta = \theta_c$ (where θ is the polar coordinate in a spherical coordinate system (R, θ, ψ)), but have a singularity at the symmetry axis $\theta = 0$ (Goldshtik 1960; Serrin 1972) and free-vortex solutions, which are perfectly regular at the symmetry axis, but cannot satisfy the no-slip condition at the conical boundary (Squire 1952; Yih *et al.* 1982; Shtern & Hunssain 1993, 1996). Perhaps there is some self-preserving swirling jet, which is closely approximated by a conically self-similar solution. Indeed, the flow inside the cone shown in Billant, Chomaz & Huerre (1998) is a natural candidate for such a flow. Even if we will never find these solutions in nature, understanding their fundamental properties remains relevant, as an intermediate step before taking on the infinitely more difficult quest of understanding fully turbulent swirling flow.

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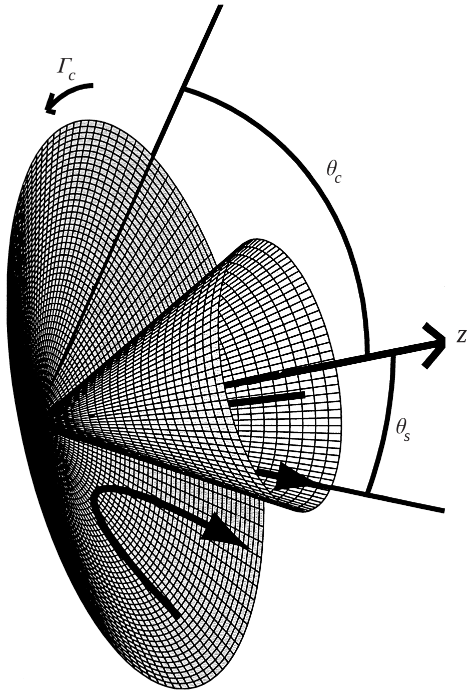

FIGURE 1. The geometry of the two-cell flow case. The symmetry axis is in the z-direction, θ_c is the angle of the bounding streamsurface and Γ_c is the circulation thereon. θ_s is the angle of the streamsurface, which separates the two cones. The thick arrows indicate the direction of the radial velocity component.

In spite of their relative simplicity many fundamental properties of the conically self-similar solutions have remained poorly understood. Recently, however, results on the existence (Stein 2000) and regularity (Stein 2001) of these solutions have been established, and, in this paper, our primary aim is to find sets of physically relevant parameters, which determine a conically self-similar free-vortex solution uniquely. In doing so, we will also see that for the parameters used by Yih *et al.* (1982) the uniqueness/non-uniqueness properties are rather intriguing.

Despite the failure of conically self-similar free-vortex solutions to satisfy all the relevant boundary conditions, these solutions exhibit features typical of swirling jets. For example, it has been shown (Yih *et al.* 1982; Shtern & Hussain 1993, 1996) that there are three kinds of conically self-similar free-vortex solutions: near axis jets, near surface jets and two-cell flows. The two-cell flow case is depicted in figure 1. Needless to say, the opening angle of the separating conical streamsurface, θ_s, is often a parameter of great significance. The near-axis jet corresponds to the limit $\theta_s = 0$ and for the surface jet we have $\theta_s = \theta_c$.

Throughout this paper, we will assume that the opening angle of the bounding conical streamsurface θ_c and the circulation $K(= 2\pi\nu\Gamma_c)$ on this streamsurface are given, where ν is the kinematic viscosity. In addition to these quantities, we must somehow control the intensity of the axial motion. A natural way to achieve this would be to specify the total axial momentum, J_z, however, the numerical and asymptotic studies in Shtern & Hussain (1993, 1996) (see also Shtern & Hussain 1999) tell us that there are combinations of values of θ_c, K and J_z for which several solutions exist. This important issue explains the observed bistability of tornadoes, but for control purposes, for example, it is highly undesirable to have several possible solutions for the same set of values of the control parameters, since that may allow the swirling jet to toggle between a desirable and an undesirable flow state at random. Furthermore, it is easier to view the non-uniqueness as a fold catastrophe, as in Shtern & Hussain (1993, 1996), if there is some way to label the solutions, i.e. some set of parameters for which uniqueness holds. Accordingly, we will study alternative quantities which describe the strength of the axial flow. The axial velocity and the pressure at the bounding streamsurface are two candidates. A third option is to use the surface radial tangential stress as suggested by Goldshtik & Shtern (1990). We will prove that for given values of θ_c and K, either of these quantities will specify a conically self-similar solution uniquely. The sole exception is in the case of a bounding plane (i.e. when $\theta_c = 90°$), in which case the pressure on the surface is always zero, and hence it contains no information on the axial flow.

We will also see that the situation is more complex if we use, along with θ_c and K, the Yih *et al.* (1982) parameter T, which was shown recently to be given by (Stein 2000)

$$T = -\frac{2}{\Gamma_c^2}\left(\frac{(p_c - p_\infty)R^2}{\rho\nu^2} - \frac{u_{Rc}R}{\nu}\right), \tag{1.1}$$

where p_c and u_{Rc} are the pressure and the radial (in spherical coordinates) velocity component at a point along the bounding conical streamsurface, R is the spherical radius of that point, p_∞ is the pressure at infinity and ρ is the density. In this case, we will prove that when $\theta_c \leqslant 90°$ (i.e. for flow inside a cone) we have uniqueness, which is not necessarily true when $\theta_c > 90°$ (i.e. for exterior flows). Another major result is that we are able to show by explicit numerical construction that there are values of $\theta_c < 0$, K and T for which non-uniqueness actually occurs.

The question of what uniquely determines a conically self-similar free-vortex solution has received virtually no attention in the literature. The major difficulty is the lack of appropriate mathematical techniques to approach questions of uniqueness for nonlinear boundary-value problems. Whereas a numerical approach may indicate the existence of a solution by explicit construction of the solution (approximately) and non-uniqueness of a problem by explicit construction of two different solutions for the same problem, the uniqueness issues are subtle in that numerics may be deceptive. Sometimes, this is due to the numerical algorithm, which may be biased to give only one of several possible solutions. Another difficulty is that non-uniqueness may occur only in small portions of parameter space, in which case it requires both luck and perseverance to detect it. For example, even though Yih *et al.* formulated their problem 17 years ago, no previous numerical study has revealed the non-uniquenesss mentioned in the previous paragraph, and the common belief among workers in the field that the solution was uniquely specified by θ_c, K and T is manifested by the comment in Shtern & Hussain (1996) that conically self-similar free-vortex solutions do 'not show any fold and non-uniqueness when different control parameters are used (Yih *et al.* 1982; Goldshtik & Shtern 1990)'. On the same note, there are some implicit assumptions that uniqueness holds in Yih *et al.* (1982). For example, it is said that 'The nondimensionalized momentum flux $M/\rho\nu^2$ is a function of the parameters *Re* [equivalent to our K] and T', which is not necessarily true in the case of non-uniqueness since the same value of K and T can give two different values of M, and hence we cannot speak of a function in the normal single-valued sense.

To the author's knowledge only one uniqueness result has been proved for conically self-similar free-vortex solutions, and it is a result in Stein (2001) which tells us that not only is such a solution real analytic outside the origin, but it is also uniquely determined by the radial velocity, the polar derivative of the circulation and the radial friction at the symmetry axis. This result is required to justify mathematically the numerical method used in Shtern & Hussain (1993, 1996), but it is of rather limited physical interest, since the quantities involved are not directly related to any of the forces driving the flow. Consequently, it is difficult to relate these parameters to important flow properties, such as the opening angle of a two-cell flow. By contrast, we will see that such relations can be derived for the quantities under consideration here. Initially, we will use the asymptotic analysis developed by Shtern & Hussain (1993, 1996) to find a relation which for given values of θ_c and K determines the surface pressure or the radial tangential stress at the surface in terms of θ_s. In addition, we will derive a converse relation, which gives θ_s provided that the values of θ_c, K and either of the surface pressure or the radial tangential stress at the surface are known. In order to obtain reasonable accuracy for these formulae at moderate values of K/ν, an alternative and more accurate asymptotic analysis will be developed and used to obtain second-order correction terms to these formulae. Fortunately, both these formulae can still be given explicitly. Numerical calculations were used to confirm that for all but large values of θ_c the presented formulae are accurate even for moderate values of K/ν.

In the next section we will formulate our problem mathematically. In § 3 we will first state our uniqueness theorem. Afterwards, we will show that in the case where $\theta_c < 0$, K and T are given, there may exist more than one solution, and we will explain this behaviour from a physical perspective. Finally, we will state and discuss a theorem, which contains asymptotic formulae relating θ_s to θ_c, to K and to either of the surface pressure or the radial tangential stress at the surface. Section 4 will be devoted entirely to the proof of the theorems.

2. Formulation of the problem

The conically self-similar solutions to the Navier–Stokes equations are defined in a conical domain $0 < \theta < \theta_c$, $r > 0$. The name conically self-similar solutions comes from the fact that the solutions are such that any quotient of two velocity components depends only on the polar angle of a properly aligned spherical coordinate system. The main physical idea behind these solutions is to seek solutions to the Navier–Stokes equations which are characterized by a streamfunction as well as by a circulation function, i.e. to seek solutions of the form:

$$\left.\begin{aligned} u_R &= -\frac{\nu\psi'(x)}{R}, \quad u_\theta = -\frac{\nu\psi(x)}{R\sin\theta}, \quad u_\phi = \frac{\nu\Gamma(x)}{R\sin\theta}, \\ p - p_\infty &= \frac{\rho\nu^2 q(x)}{R^2}, \quad \Psi = \nu R\psi(x), \quad x = \cos\theta, \end{aligned}\right\} \tag{2.1}$$

where (u_R, u_θ, u_ϕ) are velocity components in spherical coordinates, p the pressure and Ψ a streamfunction. We have also let a prime denote differentiation with respect to x.

When (2.1) is substituted into the Navier–Stokes equations, we obtain after some manipulations the following system of ODEs (Serrin 1972; Shtern & Hussain 1996):

$$(1-x^2)\psi' + 2x\psi - \tfrac{1}{2}\psi^2 = F, \tag{2.2a}$$

$$(1-x^2)F''' = 2\Gamma\Gamma', \tag{2.2b}$$

$$(1-x^2)\Gamma'' = \psi\Gamma', \tag{2.2c}$$

where F is an auxiliary function, originally introduced by Goldshtik (1960), which replaces the pressure q in the analysis.

2.1. *Boundary conditions*

In this paper, we are only interested in classical solutions to (2.2), i.e. solutions such that $\psi \in C^1((x_c, 1)) \cap C([x_c, 1]), F \in C^3((x_c, 1)) \cap C^1([x_c, 1])$ and $\Gamma \in C^2((x_c, 1)) \cap C([x_c, 1])$ which satisfy the boundary conditions at $x_c (= \cos\theta_c)$ and 1 which we are now about to specify.

The free-vortex boundary conditions are obtained by requiring that there are no flow sources on the axis, except at the origin, which implies that

$$\Gamma(1) = \psi(1) = 0. \tag{2.3}$$

For the radial velocity to be bounded outside a neighbourhood of the origin we require that

$$\lim_{x \to 1^-} |\psi'(x)| < \infty. \tag{2.4}$$

For F we specify the boundary conditions

$$F(1) = F'(1) = 0. \tag{2.5}$$

The first of these conditions follows from (2.3), (2.4) and (2.2*a*), but the second one requires in addition that $\lim_{x \to 1^-} (1 - x^2)\psi''(x) = 0$, which physically means that there is no line force acting along $x = 1$.

We also assume that the swirling flow is driven by a constant circulation along some fixed conical streamsurface $x = x_c$ which implies that

$$\psi(x_c) = 0, \quad \Gamma(x_c) = \Gamma_c. \tag{2.6}$$

Since the system of equations (2.2) is symmetric with respect to the sign of Γ, the sign is immaterial, and we will henceforth assume that $\Gamma_c \geqslant 0$.

Before we conclude this section, we must define formally the parameters to be used to control the axial flow. The first such parameter mentioned in § 1, is the radial velocity distribution at the streamsurface, which must be of the form $1/r$ to be compatible with conical self-similarity and is given by $\psi'(x_c)$. The surface radial tangential stress per unit length of the symmetry axis, which also must be of the form $1/r$ to be compatible with conical self-similarity, is determined by $\psi''(x_c)$, as seen from Goldshtik & Shtern (1990)

$$\int_0^{2\pi} -\tau_{r\theta} r \sin\theta \cos\theta \, \mathrm{d}\phi = 2\pi x_c (1 - x_c^2)\psi''(x_c)\rho \nu^2 r^{-1}. \tag{2.7}$$

Another alternative is to specify the pressure at the bounding streamsurface, which must be of the form $1/r^2$, and is given by (see e.g. Shtern & Hussain 1996, equation (8)) $q(x_c) = -x_c\psi''(x_c)$. Consequently, $\psi''(x_c)$ defines both the pressure at the surface and the surface radial tangential stress. In order to obtain the parameter T used by Yih *et al.* (1982) we integrate (2.2*b*) three times. After the enforcement of the conditions (2.5) this yields

$$F(x) = -(1 - x)^2 \int_{x_c}^{x} \frac{t\Gamma^2 \mathrm{d}t}{(1 - t^2)^2} - x \int_x^1 \frac{\Gamma^2 \mathrm{d}t}{(1 + t)^2} - \tfrac{1}{2} T \Gamma_c^2 (1 - x)^2, \tag{2.8}$$

where T is an arbitrary parameter. In Stein (2000) it was shown that for T defined in this way we have that:

$$T = \frac{1}{1-x_c^2} - \frac{F''(x_c)}{\Gamma_c^2} = -\frac{2}{\Gamma_c^2}(q(x_c) + \psi'(x_c)), \tag{2.9}$$

which is equivalent to (1.1).

3. Results and discussion

We begin by presenting the main result in this paper.

Theorem 1. *Suppose that $x_c \in (-1, 1)$ then a conically self-similar free-vortex solution to the Navier–Stokes equations is uniquely determined by x_c, Γ_c and one of* (*a*) *$\psi'(x_c)$,* (*b*) *$\psi''(x_c)$ or in case $x_c \geqslant 0$* (*c*) *T.*

Remark 1. The theorem does not necessarily hold in case $x_c = -1$.

Remark 2. The theorem contains no statement on the continuous dependence of parameters, which is an important problem still left open. However, in the parameter ranges studied in this paper, the numerical calculations performed do not hint at any discontinuities.

The proof of this theorem is fairly long and technical and will be deferred until the next section. In § 3.1 we will see that when $x_c < 0$ then there may exist more than one conically self-similar free-vortex solution with the same values of x_c, Γ_c and T. In § 3.2 we will derive formulae for the two-cell flow solutions, which for given values of x_c and Γ_c allow us to calculate the angle of the separating cone for the two-cell solutions, $x_s (= \cos\theta_s)$ in terms of $\psi''(x_c)$ or vice versa.

3.1. *Non-uniqueness for the T-problem when $x_c < 0$*

When $x_c < 0$, the proof of theorem 1(*c*) fails in only one step. Clearly, the failure could be due to a weakness of the method of proof, and uniqueness could still hold. For some indication of the validity of this hypothesis, we would like to make some numerical experiments. To perform such experiments, we must have some feeling for what kind of non-uniqueness we could possibly expect, and it seems that the most likely non-uniqueness is that for the same combination of values of $x_c < 0$, Γ_c and T there are two two-cell flow solutions with different values of x_s. In the case where $x_c = -0.15$ and $\Gamma_c = 30$, we chose various values of x_s, and used a numerical algorithm, which is an obvious modification of the one described in Shtern & Hussain (1996, p. 33) to solve our system numerically. For each value of x_s, we calculated the corresponding value of T, and the results are shown in figure 2. From this figure, we see that T is not a monotone function of x_s, and there are several values of T for which at least two different solutions exist. For example, in the case when $T = 0.014$ we have one solution with $x_s \approx 0.915$ and one with $x_s \approx 0.355$. These solutions were calculated using the algorithm in Shtern & Hussain (1993, 1996) and are shown in figure 3.

These somewhat surprising results for the T problem call for a physical explanation. From (2.9) it is seen that T is directly related to the pressure and the radial velocity at the conical streamsurface, except when $x_c = 0$, in which case $q(0) = 0$. As the opening angle of the annular jet in a two-cell flow decreases, i.e. as x_s increases, we know from Shtern & Hussain (1996) that, at least asymptotically for large Γ_c, the magnitude of the radial velocity at the bounding streamsurface $x = x_c$ increases, and hence since $v_r = -\nu\psi'/r$, $\psi'(x_c)$ increases with increasing x_s.

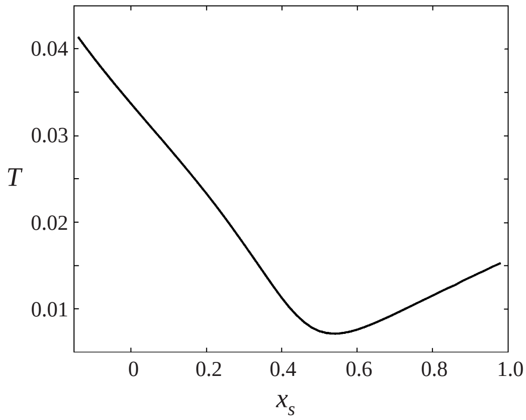

FIGURE 2. The variation of T as a function of x_s when $\Gamma_c = 30$ and $x_c = -0.15$.

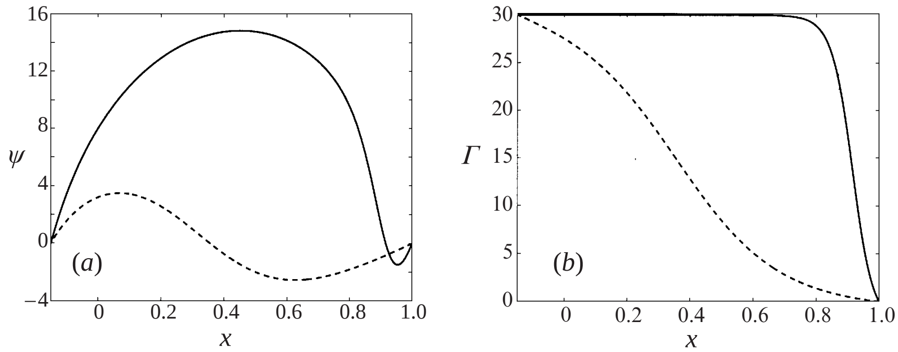

FIGURE 3. Two different numerical solutions with $x_c = -0.15$, $\Gamma_c = 30$ and $T = 0.014$. ——, solution with $x_s \approx 0.915$, - - - -, solution with $x_s \approx 0.355$.

On the other hand, the asymptotic value of the pressure at the bounding streamsurface can be obtained from the asymptotic solutions found by Shtern & Hussain (1996). (We present the formulae here since there are some misprints in the corresponding formulae in Shtern & Hussain (1996).)

$$q = \begin{cases} -\Gamma_c^2 \dfrac{(x_s - x_c^2)x - [2x_s - (1+x_s)x_c]x_c}{(1+x_s)(1-x_c)^2(1-x^2)} + o(\Gamma_c^2) & \text{for } x_c \leqslant x \leqslant x_s, \\ -\Gamma_c^2 \dfrac{(x_s - x_c)^2}{(1+x)(1-x_c)^2(1-x_s^2)} + o(\Gamma_c^2) & \text{for } x_s \leqslant x \leqslant 1, \end{cases} \tag{3.1}$$

Hence, we have

$$\frac{\mathrm{d}}{\mathrm{d}x_s}(q(x_c)) = \frac{x_c \Gamma_c}{(1+x_s)^2(1-x_c)^2}. \tag{3.2}$$

Consequently, $q(x_c)$ increases with increasing x_s when $x_c > 0$, and it decreases with increasing x_s when $x_c < 0$. To summarize, we have shown, at least asymptotically when Γ_c goes to infinity, that when $x_c > 0$, $q(x_c)$ and $\psi'(x_c)$ both increase as x_s increases, whereas when $x_c < 0$ they move in opposite directions as x_s changes. This allows us to conclude that T is a monotone (decreasing) function of x_s when $x_c \geqslant 0$, but we cannot draw any such conclusion when $x_c < 0$.

To conclude this section we will comment on the numerical algorithm presented in Yih *et al.* (1982). The idea is to start with $\Gamma_0 \equiv \Gamma_c$ and then iteratively calculate F_1, ψ_1, Γ_1, F_2 etc. from the formulae in (2.2). In practice, the method seems to converge for values of x_c, Γ_c and T for which a solution exists. Here, it is of interest to study the performance of this algorithm in a case when the solution is not unique. To this end, this method was implemented for the case mentioned above when $x_c = -0.15$, $\Gamma_c = 30$ and $T = 0.014$, and it was found that this method seemed to converge towards the solution with $x_s \approx 0.355$. Hence, to detect non-uniqueness with this numerical method we would have to run the algorithm with several different starting functions Γ_0. This shows the danger of drawing conclusions about uniqueness exclusively from numerical simulations.

3.2. *Asymptotic relations between $\psi''(x_c)$ and x_s for given values of Γ_c and x_c*

Theorem 1 tells us that for any values of Γ_c, x_c and $\psi''(x_c)$ for which a two-cell solution exists, there corresponds a unique value of x_s, and in principle we can calculate x_s numerically. In control terms, this can be expressed as follows. Suppose that for given values of x_c and Γ_c we want to obtain a particular value of x_s, and that we have found a value of $\psi''(x_c)$ which gives us this optimal value, then we can be sure that it does not correspond to any other solution. At this point, it should be stressed that we cannot say that this is the only optimal value, since we have not proved that a conically self-similar solution is uniquely determined by x_c, Γ_c and x_s. (Most of the numerical computations in this paper are calculated by a method of Shtern & Hussain (1996) for which these values are given and tentative values at x_s are adjusted to satisfy the boundary conditions at x_c and 1. In the event that uniqueness does not hold in this case the numerical results below will only be representative for solutions approaching Shtern & Hussain's asymptotic solution.) Nevertheless, it is possible to calculate numerically values of $\psi''(x_c)$ corresponding to such optimal solutions. However, as Γ_c increases, the equations become increasingly singular, and hence more difficult to solve numerically. Accordingly, we would like to have asymptotic formulae which express the relation between these quantities in the high-Γ_c limit. Fortunately, such formulae can be obtained from the asymptotic analysis of Shtern & Hussain (1996), by substituting (3.1) into $q(x_c) = x_c\psi''(x_c)$ to obtain

$$\psi''(x_c) = \psi''(x_c)^{a1} + o(\Gamma_c^2), \tag{3.3}$$

$$\psi''(x_c)^{a1} = -\frac{\Gamma_c^2(x_s - x_c)}{(1-x_c)^2(1+x_c)(1+x_s)}. \tag{3.4}$$

Suppose that $\psi''(x_c) \approx \psi''(x_c)^{a1}$. We can then expect x_s^*, obtained by inverting the formula for $\psi''(x_c)^{a1}$, to yield a reasonable approximation for x_s in terms of $\psi''(x_c)$. Indeed, by simple algebra we have

$$x_s^* = \frac{x_c\Gamma_c^2 - (1-x_c)^2(1+x_c)\psi''(x_c)}{\Gamma_c^2 + (1-x_c)^2(1+x_c)\psi''(x_c)}. \tag{3.5}$$

Since $q(x_c) = -x_c\psi''(x_c)$ and $q \propto v^{-2}$, we find that both the nominator and the denominator in (3.5) are proportional to v^{-2}. Consequently, the predicted value of x_s does not depend on v as long as v is small enough to secure the validity of the asymptotic approximation. Hence, this formula exhibits Reynolds-number invariance for large Reynolds numbers.

To study the speed of convergence towards these asymptotic formulae, numerical

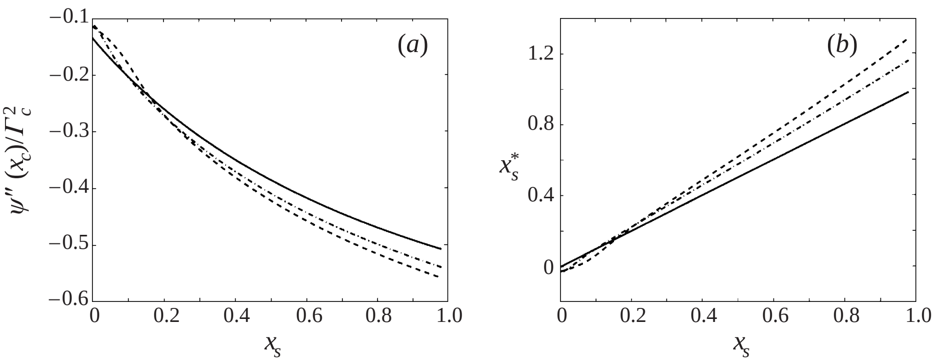

FIGURE 4. A comparison between (*a*) $\psi''(x_c)^{a1}/\Gamma_c^2$ and (*b*) x_s^* and their respective numerically computed true values when $x_c = -0.15$. (*a*) ——, $\psi''(x_c)^{a1}/\Gamma_c^2$; - - - -, numerically computed values of $\psi''(x_c)/\Gamma_c^2$ for $\Gamma_c = 150$; –·–, numerically computerd values of $\psi''(x_c)/\Gamma_c^2$ for $\Gamma_c = 300$; (*b*) ——, an ideal prediction curve, i.e. a line with slope 1; – – –, x_s^* obtained from the numerically obtained values of $\psi''(x_c)$ for $\Gamma_c = 150$, –·–, x_s^* obtained similarly for $\Gamma_c = 300$.

computations were performed for several values of x_s and Γ_c, and some of these results are presented in figure 4. From this figure, we see that even when $\Gamma_c = 300$ the approximations differ from the true values by more than 15%, and hence we conclude that in this case much higher values of Γ_c than 300 are required to obtain reasonable accuracy for the asymptotic formulae. Yet when $\Gamma_c = 300$, $x_c = -0.15$ and x_s is not too small, the conically self-similar solutions are close to the asymptotic ones for most of the domain $[x_c, 1]$. Specifically, Γ is close to the asymptotic function $\Gamma = \Gamma_c H(x_s - x)$, where H is a Heaviside function. Clearly, this makes numerical computations rather difficult for larger values of Γ_c owing to the large values of the gradients.

In order to derive more accurate formulae, we must understand where the discrepancies in figure 4 arise. Essentially, Shtern & Hussain's asymptotic analysis starts by assuming that Γ is given by the asymptotic function above. This function is then used to calculate F, which is given by second-degree polynomials in each of the domains $[x_c, x_s]$ and $[x_s, 1]$, to which we can add an arbitrary term of the form $C(1-x)^2$ for any C. This term gives a contribution to $\psi''(x_c)$ given by $-2C/(1+x_c)$, and hence an accurate determination of this value is very important for the present analysis. In order to obtain the solutions of highest order in Γ_c, Shtern & Hussain removed the linear terms in (2.2*a*) and calculated ψ from F by simple algebraic operations, and then chose C in such a way that the boundary condition $\psi(x_c) = 0$ could be satisfied.

As a consequence of their choice of C, Shtern & Hussain could not satisfy the condition $\psi(x_s) = 0$, even though it was assumed to hold in the beginning of the analysis. This and other shortcomings were remedied with the use of inner solutions around x_s and x_c. However, the inner solutions in Shtern & Hussain (1996) are not directly applicable in our case since they assume that F is constant in the boundary layer around $x = x_c$, which yields no correction to $F'(x_c)$ or $\psi''(x_c)$.

Instead of introducing more complex inner solutions, a different approach will be used to obtain formulae valid in the entire domain. To this end, we will assume that Γ is given by the asymptotic function above, and calculate F as in Shtern & Hussain (1996), but we will retain the arbitrary term $C(1-x)^2$, where, for convenience, we define C in such a way that for $C = 0$ we obtain the same functions that Shtern & Hussain used. For given values of x_c, Γ_c and x_s, our problem is to find a C such that both of the boundary-value problems

$$\left.\begin{aligned}&(1-x^2)\psi_1' + 2x\psi_1 - \tfrac{1}{2}\psi_1^2 = F_1,\\ &F_1 = \Gamma_c^2\left\{-(x-x_c)p_1(x) + C(1-x)^2\right\},\\ &p_1(x) = \frac{2x_s - (1+x_s)x_c - (1+x_s-2x_c)x}{2(1+x_s)(1-x_c)^2},\\ &\psi_1(x_c) = 0, \quad \psi_1(x_s) = 0,\end{aligned}\right\} \tag{3.6}$$

$$\left.\begin{aligned}&(1-x^2)\psi_2' + 2x\psi_2 - \tfrac{1}{2}\psi_2^2 = F_2,\\ &F_2 = \Gamma_c^2(1-x)^2\left\{-\frac{(x_s-x_c)^2}{2(1-x_s^2)(1-x_c)^2} + C\right\},\\ &\psi_2(x_s) = 0, \quad \psi_2(1) = 0,\end{aligned}\right\} \tag{3.7}$$

have bounded solutions, i.e. we have an eigenvalue problem, where the additional freedom in the choice of C is to be used to satisfy all the boundary conditions. It can be seen that there exists some C^* such that the second of the boundary-value problems can be solved for any $C < C^*$. This follows from the fact that it is solved by a Squire–Potsch solution (see e.g. Potsch 1981 or Stein 2000). Hence, we must only solve the first eigenvalue problem. To this end, let us make the substitution $\psi = -2(1-x^2)U'/U$ which yields

$$-U'' + \frac{\Gamma_c^2(x-x_c)p_1(x)}{2(1-x^2)^2}U = \frac{C\Gamma_c^2}{2(1+x)^2}U, \tag{3.8}$$

with the boundary conditions $U'(x_c) = U'(x_s) = 0$, and the additional condition that $U \neq 0$ for $x \in [x_c, x_s]$. This is a Sturm–Liouville problem and hence we know that there exists an increasing sequence of non-negative eigenvalues, C_i, $i = 0, 1, 2, \dots$. such that, for $C = C_i$, (3.8) has an eigenfunction with i zeros in $[x_c, x_s]$. The solution we seek is clearly the one with no zeros, which corresponds to the lowest eigenvalue $C = C_0$. For Sturm–Liouville problems, there are several algorithms to facilitate numerical computation of the eigenvalues and eigenfunctions. Consequently, we can, for given values of Γ_c, x_c and x_s, compute a correction term to $\psi''(x_c)^{a1}$, which would greatly improve the accuracy of this estimate. However, with no analytical expression at hand, we are at a loss in trying to invert the formula to obtain a more accurate formula for x_s^* in terms of Γ_c, x_c and $\psi''(x_c)$. Fortunately, however, we can derive an asymptotic expression for C valid for large Γ_c given by the following theorem.

THEOREM 2. *For any Γ_c, x_c and x_s, the lowest eigenvalue $C(= C_0)$ to (3.6) is given by*

$$C = \frac{2^{1/3}(-\alpha_1')}{(1-x_c)^2}\left(\frac{(x_s-x_c)(1+x_c)}{1+x_s}\right)^{2/3}\Gamma_c^{-2/3} + o(\Gamma_c^{-2/3}), \tag{3.9}$$

for large values of Γ_c. Here $\alpha_1' \approx -1.01879297$ is the first zero of the first derivative of the Airy function, Ai′.

The proof of the theorem uses the asymptotic solutions for second-order ordinary differential equations with a (first-order) turning (transition) point, and is deferred to the next section.

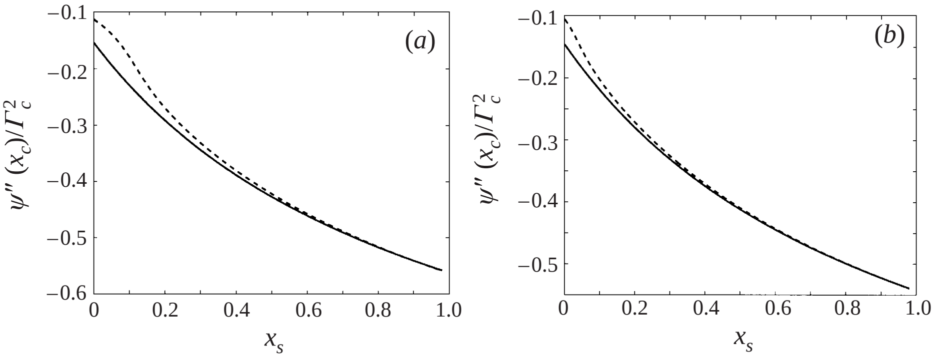

FIGURE 5. A comparison between ——, $\psi''(x_c)^{a2}/\Gamma_c^2$ and - - - -, true values of $\psi''(x_c)/\Gamma_c^2$ obtained numerically. In all numerical computations $x_c = -0.15$, (*a*) $\Gamma_c = 150$, (*b*) $\Gamma_c = 300$.

This theorem yields a refined asymptotic approximation for $\psi''(x_c)$

$$\psi''(x_c)^{a2} = -\frac{f(\Gamma_c^2(x_s - x_c)/(1 + x_s), 2^{4/3}(-\alpha_1')(1 + x_c)^{2/3})}{(1 - x_c)^2(1 + x_c)}, \tag{3.10}$$

$$f(x, a) = x + ax^{2/3}. \tag{3.11}$$

In fact, it is possible to invert the formula (3.10) explicitly. To that end, let $z = (1 - x_c)^2(1 + x_c)\psi''(x_c)$ and let $f^{-1}(x, a)$ denote inverse function of $f(x, a)$ with respect to x. We then have

$$x_s^\dagger = \frac{x_c\Gamma_c^2 - f^{-1}(z, a)}{\Gamma_c^2 + f^{-1}(z, a)}, \tag{3.12}$$

but using Cardano's formula we can calculate $f^{-1}(z, a)$ explicitly to obtain

$$f^{-1}(z, a) = \left\{-\tfrac{1}{3}a + \left(\tfrac{1}{2}z - \tfrac{1}{27}a^3 + D\right)^{1/3} + \left(\tfrac{1}{2}z - \tfrac{1}{27}a^3 - D\right)^{1/3}\right\}^3, \tag{3.13}$$

where

$$D = \left(\tfrac{1}{4}z^2 - \tfrac{1}{27}a^3 z\right)^{1/2}. \tag{3.14}$$

(There are two other inverse formulae, but these are complex and hence of no interest here.)

In figures 5 and 6, the corrected asymptotic formulae (3.10) and (3.12) are compared with numerically obtained true values. These figures show that unless x_s is small, then the refined asymptotic formulae are accurate within a few per cent even for $\Gamma_c = 150$. This is clearly a significant improvement compared to the situation presented in figure 4. Nevertheless, it is still true that, for sufficiently large Γ_c, the highest-order formulae, which were presented in the beginning of this subsection, are still valid, as is the conclusion of the Reynolds-number invariance of x_s^* and $x_s^\dagger$ for sufficiently large Reynolds numbers.

When x_s is small, the refined asymptotic formulae are no longer as accurate, but this is due to the fact that Γ is no longer close to the asymptotic function as it was assumed to be. This, however, means that the system is fairly simple to solve numerically, so we may obtain relations between x_s and $\psi''(x_c)$ that way. Indeed, it seems to be a general fact that the refined asymptotic formulae are accurate whenever numerical solutions are not readily available. Hence, if the approaches are combined, for any Γ_c and x_c, we can obtain the relations between x_s and $\psi''(x_c)$, which are accurate to

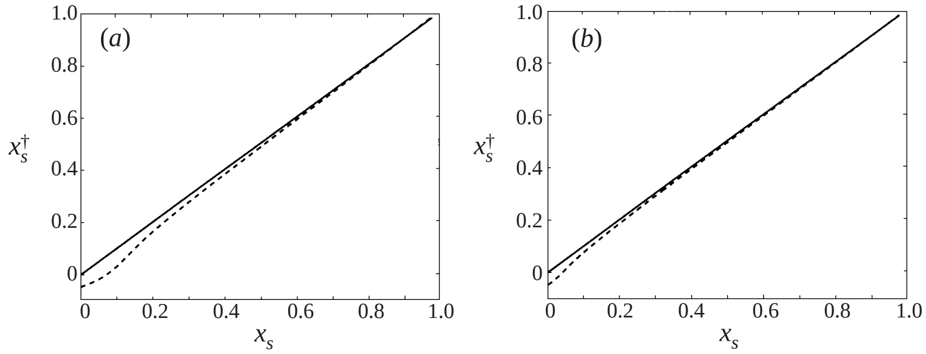

FIGURE 6. A comparison between - - - -, $x_s^\dagger$ calculated from numerically obtained values of $\psi''(x_c)$ and ——, the ideal curve given by a straight line through the origin with slope 1. In all numerical computations $x_c = -0.15$, (*a*) $\Gamma_c = 150$, (*b*) $\Gamma_c = 300$.

within a few per cent, and thus we can with reasonable accuracy determine the angle of the separating cone in a two-cell flow solution, exclusively from knowledge of the flow at the bounding streamsurface.

Before concluding this section, it should be said that we could equally well have derived formulae relating $\psi'(x_c)$ and x_s to one another. Indeed, with C chosen as in theorem 2 it is readily seen that

$$\psi'(x_c) = \frac{2^{1/3}(-\alpha_1')}{1-x_c^2}\left(\frac{(x_s-x_c)(1+x_c)}{1+x_s}\right)^{2/3}\Gamma_c^{4/3} + o(\Gamma_c^{4/3}), \tag{3.15}$$

which we can invert. This is exactly the same solution as the one obtained in Shtern & Hussain (1996)(except for a slight inaccuracy in their formula), but with the additional bonus that we have shown that their constant 1.2836, obtained numerically, is only an approximation of $2^{1/3}(-\alpha_1')$, which shows the origin of the constant. There is, however, an important difference between the formulae obtained using $\psi'(x_c)$ and those obtained using $\psi''(x_c)$, namely that in the former case the asymptotically leading terms come from the boundary layer around $x = x_c$, whereas in the latter case they come from the outer solution. Hence, we can expect that formulae using $\psi'(x_c)$ are more sensitive to the exact nature of the boundary layer.

4. Proofs of the main theorems

This section will be entirely devoted to proving theorems 1 and 2, and we will begin by stating some known auxiliary results.

The auxiliary function F in (2.2*b*) was originally introduced by Goldshtik (1960). It was integrated once by Sozou (1992) to yield

$$(1-x^2)F'' + 2xF' - 2F = \Gamma^2. \tag{4.1}$$

The major technical tool required to prove theorem 1 is the following result proved in Stein (2001)

THEOREM 3. (*Real analyticity*)
For any solution to (2.2a), (2.2c) and (4.1) with $x_c \neq -1$ *such that* $\psi \in C^1([x_c, 1])$, $F \in C^2([x_c, 1)$ *and* $\Gamma \in C^2([x_c, 1])$, *and which satisfies the boundary conditions* (2.3)–(2.6), *there exists a domain* $\Omega \subset \mathbf{C}$ *which contains* $[x_c, 1)$ *as well as a ball* $B(1, r)$ *for*

some $r > 0$, *such that* ψ, F *and* Γ *in fact belong to* $A(\Omega)$. *Here* $A(\Phi)$ *denotes the class of analytic (holomorphic) functions in* Φ, *and* $B(a, r)$ *is the disc in the complex plane with centre at* a *and radius* r.

In addition, we will on some occasions use the simple observation that since ψ is real analytic and satisfies (2.3) and (2.4) we may integrate (2.2*c*) to obtain

$$\Gamma'(x) = \Gamma'(x_c) \exp\left(\int_{x_c}^{x} \frac{\psi(z)}{1 - z^2} \mathrm{d}z\right). \tag{4.2}$$

From this, it is evident that Γ' is monotonous in $[x_c, 1]$ and thus if $\Gamma(x_c) = \Gamma_c > 0$ and $\Gamma(1) = 0$ it is clear that Γ' is negative in $[x_c, 1]$ and hence that Γ is positive in $[x_c, 1]$.

4.1. *The proof of theorem 1*

To begin with, let us transform our equations to the form used by, for example, Serrin (1972) or Yin *et al.* (1982). To this end, let us make the substitutions

$$f(x) = -\frac{\psi(x)}{2(1 - x^2)}, \quad \Omega(x) = \frac{\Gamma(x)}{\Gamma_c}, \quad G(x) = -\frac{2}{\Gamma_c^2} F(x), \tag{4.3}$$

to obtain the system

$$f' + f^2 = \left(\frac{\Gamma_c}{2}\right)^2 \frac{G(x)}{(1 - x^2)^2}, \tag{4.4a}$$

$$(1 - x^2)G'' + 2xG' - 2G = -2\Omega^2, \tag{4.4b}$$

$$\Omega'' + 2f\Omega' = 0. \tag{4.4c}$$

We remark that f is real analytic, because of (2.3)–(2.4) and the real analyticity of ψ. The equation (4.4*b*) can be integrated to obtain

$$G(x) = 2(1 - x)^2 \int_{x_c}^{x} \frac{t\Omega^2 \mathrm{d}t}{(1 - t^2)^2} + 2x \int_{x}^{1} \frac{\Omega^2 \mathrm{d}t}{(1 + t)^2} + T(1 - x)^2, \tag{4.5}$$

with T given by (2.9). A solution to (4.4) satisfies the boundary conditions

$$f(x_c) = 0, \tag{4.6a}$$

$$\lim_{x \to 1^-} |f(x)| < \infty, \quad \lim_{x \to 1^-} |(1 - x)f'(x)| < \infty, \tag{4.6b}$$

$$\Omega(x_c) = 1, \quad \Omega(1) = 0, \tag{4.6c}$$

if and only if the boundary conditions for our original problem (2.3)–(2.6) are satisfied. Furthermore, we must transform our additional conditions to this form. In case (*a*) we specified $\psi'(x_c)$, but the definition of f tells us that once x_c is fixed this is equivalent to specifying $f'(x_c)$. However, from (4.4*a*) and (4.6*a*) it follows that if, in addition, Γ_c is given, it also determines $G(x_c)$. In case (*b*), where we specify $\psi''(x_c)$, we must proceed differently. Now we differentiate (2.2*a*) to obtain

$$(1 - x^2)\psi'' + 2\psi - \psi\psi' = F'. \tag{4.7}$$

Thus, if we fix $\psi''(x_c)$ and x_c we also specify $F'(x_c)$, which means that if, in addition, we are given the value of Γ_c, we are given $G'(x_c)$. (It is not true that we have fixed $f''(x_c)$ though.) For the case (*c*) we specify T in (4.5), which according to (2.9) and (4.3) implies that $G''(x_c)$ is fixed if T and x_c is.

Proof of Theorem 1.

To prove Theorem 1 we will assume that there are two conically self-similar free-vortex solutions, (f_1, G_1, Ω_1) and (f_2, G_2, Ω_2), having the same values of x_c, Γ_c and either (a) $G(x_c)$, (b) $G'(x_c)$ or (c) T, and form their differences. We will then show that the difference of the two Ωs must have an infinite number of zeros in the compact interval $[x_c, 1]$, and hence that the set of zeros must have a limit point. However, theorem 3 tells us that a conically self-similar free-vortex solution to the Navier–Stokes equations is real analytic, and consequently the difference of two such solutions is also real analytic. Therefore, the uniqueness theorem for real analytic functions implies that the two Ωs must be the same, which in turn implies that the two Gs and the two fs must coincide.

Accordingly, let us define

$$\mathscr{F} = f_1 - f_2, \quad \mathscr{G} = G_1 - G_2, \quad \mathscr{W} = \Omega_1 - \Omega_2, \quad \mathscr{T} = T_1 - T_2, \tag{4.8}$$

where T_1 and T_2 are the values of T of the two different solutions, which are the same in case (c) but not necessarily in the other two cases. These quantities clearly satisfy

$$\mathscr{F}' + (f_1 + f_2)\mathscr{F} = \frac{1}{4}\Gamma_c^2 \frac{\mathscr{G}}{(1-x^2)^2}, \tag{4.9a}$$

$$\mathscr{G} = 2(1-x)^2 \int_{x_c}^{x} \frac{t(\Omega_1 + \Omega_2)\mathscr{W}}{(1-t^2)^2}\mathrm{d}t + 2x \int_x^1 \frac{(\Omega_1 + \Omega_2)\mathscr{W}}{(1+t)^2}\mathrm{d}t + \mathscr{T}(1-x)^2, \tag{4.9b}$$

$$\mathscr{W}'' + (f_1 + f_2)\mathscr{W}' + (\Omega_1' + \Omega_2')\mathscr{F} = 0, \tag{4.9c}$$

as well as the boundary conditions

$$\mathscr{F}(x_c) = \mathscr{W}(x_c) = \mathscr{W}(1) = \mathscr{G}(1) = \mathscr{G}'(1) = 0. \tag{4.10}$$

In addition to these boundary conditions, we have

$$\mathscr{G}(x_c) = 0 \quad \text{for case } (a), \tag{4.11a}$$

$$\mathscr{G}'(x_c) = 0 \quad \text{for case } (b), \tag{4.11b}$$

$$\mathscr{G}''(x_c) = 0 \quad \text{for case } (c). \tag{4.11c}$$

For case (c) we could equivalently have expressed this as

$$x_c\mathscr{G}'(x_c) = \mathscr{G}(x_c). \tag{4.12}$$

Now let $Z(\mathscr{W})$ denote the number of zeros of $\mathscr{W}$ in the open interval $(x_c, 1)$, and let us analogously define $Z(\mathscr{G})$ and $Z(\mathscr{F})$. The key element of this proof is the establishment of the following proposition.

Proposition 1. *For cases (a) and (b), and if $x_c \geqslant 0$ for case (c) as well, $Z(\mathscr{W})$ cannot be finite.*

Let us for a moment assume that this proposition has been proved. This implies that the set of zeros for $\mathscr{W}$ must have a limit point in $[x_c, 1]$. Let A denote the (thus non-empty) set of limit points of this set. Trivially, A must be closed in $[x_c, 1]$. On the other hand, theorem 3 establishes that both Ω_1 and Ω_2 and hence $\mathscr{W}$ are real analytic in $[x_c, 1]$. Since a real analytic function is given by its Taylor series, the function is either identically zero on the component or there exists a punctured neighbourhood around each zero where the function is non-zero. Thus, every point $a \in A$ is in the interior of A and hence A is open. Since $[x_c, 1]$ is connected and A is non-empty we must thus have that $A = [x_c, 1]$, which is equivalent to saying that $\mathscr{W} \equiv 0$.

If $\mathscr{W} \equiv 0$, it immediately follows from (4.9*b*) and (4.11) that $\mathscr{G} \equiv 0$. When this is substituted into (4.9*a*) we obtain

$$\mathscr{F}(x) \exp\left(\int_{x_c}^{x} (f_1(t) + f_2(t))\mathrm{d}t\right) = C. \tag{4.13}$$

However, the condition $\mathscr{F}(x_c) = 0$ requires that $C = 0$ and thus that $\mathscr{F} \equiv 0$. This concludes the proof of the theorem, provided that we can prove proposition 1. □

The basic idea in the proof of proposition 1 is to assume that $Z(\mathscr{W}) = n$, where n is an arbitrary non-negative integer, and then to use properties of our system to show that this implies that $Z(\mathscr{W}) \geqslant n + 1$, which establishes the proposition by contradiction. However, zeros of even multiplicity where $\mathscr{W}$ does not change sign will cause difficulties for our argument, and therefore we will only consider zeros of odd multiplicity where $\mathscr{W}$ does change sign. We will denote the number of zeros of odd multiplicity of $\mathscr{W}$ in the open interval $(x_c, 1)$ by $O_Z(\mathscr{W})$, and the same notation will be used for functions other than $\mathscr{W}$. Of course, $Z(\mathscr{W}) \geqslant O_Z(\mathscr{W})$ and thus proving that $O_Z(\mathscr{W})$ cannot be finite establishes the same conclusion for $Z(\mathscr{W})$.

The tool we will use most frequently is a simple fact in calculus, which is often termed Rolle's theorem, and which tells us that if a continuously differentiable function h is zero at two points a and b (the multiplicity of the zeros is immaterial) then either its derivative changes sign at some zero of odd multiplicity, strictly between a and b, or $h \equiv 0$. If the second alternative would hold anywhere in our subsequent argument we could immediately conclude that the concerned function would be identically zero, by invoking the uniqueness theorem for real analytic functions. This would immediately establish our theorem, and therefore for reasons of brevity this possibility will not be mentioned when Rolle's theorem is invoked below.

It turns out that one of the most difficult tasks of the proof is to obtain a lower bound of $O_Z(\mathscr{W})$ in terms of $O_Z(\mathscr{G})$. To accomplish this we introduce the auxiliary function

$$\mathscr{H} = \frac{\mathscr{G}}{(1-x)^2}, \tag{4.14}$$

which is similar to the one used by Yih *et al.* (1982). However, we will only be concerned with its derivative

$$\mathscr{H}' = \frac{(1-x)\mathscr{G}' + 2\mathscr{G}}{(1-x)^3}. \tag{4.15}$$

In fact, we can now prove the following lemma.

Lemma 1. *If $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{W}$ satisfy (4.9)–(4.10) and $\mathscr{H}'$ is as in (4.15) we have the following inequalities*

$$O_Z(\mathscr{G}) \geqslant O_Z(\mathscr{F}) \geqslant O_Z(\mathscr{W}) \geqslant O_Z(\mathscr{H}'). \tag{4.16}$$

Proof. The inequalities will be proved one at a time.

Since $\mathscr{F}(x_c) = 0$, we have that $\mathscr{F}$ has at least $O_Z(\mathscr{F}) + 1$ zeros in $[x_c, 1)$. Hence, to prove the first inequality in lemma 1, it suffices to prove that $\mathscr{G}$ changes sign at least once strictly between adjacent zeros of $\mathscr{F}$. Suppose to the contrary that there are two points x_1 and x_2 such that $\mathscr{F}(x_1) = \mathscr{F}(x_2) = 0$, and such that $\mathscr{G} \geqslant 0$ in (x_1, x_2) with equality, at most, at a finite number of points. (If $\mathscr{G} \leqslant 0$ we may of course interchange the role of the two solutions making it up.) If we now apply a Riccati transform to $f_1 = U_1'/U_1$ and to $f_2 = U_2'/U_2$ in (4.8), equation (4.4a) becomes

$$U_1'' - \frac{\Gamma_c^2}{2}\frac{G_1(x)}{(1-x^2)^2}U_1 = 0, \tag{4.17a}$$

$$U_2'' - \frac{\Gamma_c^2}{2}\frac{G_2(x)}{(1-x^2)^2}U_2 = 0, \tag{4.17b}$$

where $U_i'(x_1) = 0$ for $i = 1, 2$. Since both f_1 and f_2 are real analytic, and hence non-singular, $U_i(x) \neq 0$ for $i = 1, 2$ and $x \in [x_1, 1]$. Therefore by Sturm's second comparison theorem, see for example (Ince 1956, p. 229), we have

$$f_1 = \frac{U_1'}{U_1} > \frac{U_2'}{U_2} = f_2, \tag{4.18}$$

when $x \in (x_1, 1]$. This contradicts the assumption that $\mathscr{F}(x_2) = f_1(x_2) - f_2(x_2) = 0$. Hence, $\mathscr{G}$ must change sign between adjacent zeros of $\mathscr{F}$, and thus we have

$$O_Z(\mathscr{G}) \geqslant Z(\mathscr{F}) \geqslant O_Z(\mathscr{F}) \tag{4.19}$$

which establishes the first inequality in lemma 1.

Since $\mathscr{W}(x_c) = \mathscr{W}(1) = 0$, Rolle's theorem implies that

$$O_Z(\mathscr{W}') \geqslant O_Z(\mathscr{W}) + 1. \tag{4.20}$$

At zeros of multiplicity one of $\mathscr{W}'$, equation (4.9c) becomes

$$\mathscr{W}'' = -(\Omega_1' + \Omega_2')\mathscr{F}. \tag{4.21}$$

In the last paragraph of the previous subsection we showed that $\Omega_1' < 0$ and $\Omega_2' < 0$ in $[x_c, 1]$. Hence, the coefficient in front of $\mathscr{F}$ in (4.21) is positive at every single zero of $\mathscr{W}'$, and hence $\mathscr{F}$ has the same sign as $\mathscr{W}''$. More generally, at zeros of odd multiplicity p of $\mathscr{W}'$, we have that $\mathscr{W}''$ has a zero of even multiplicity $p-1$ at that point. Equation (4.9c) and the above remark that the coefficient in front of $\mathscr{F}$ is positive together imply that $\mathscr{F}$ has a zero of even multiplicity $p-1$ at that point. Therefore, $\mathscr{F}$ and $\mathscr{W}''$ do not change sign at zeros of odd multiplicity of $\mathscr{W}'$, and hence we may take a sufficiently small punctured neighbourhood around each of these zeros in which $\mathscr{F}$ has constant sign, which is the same as that of $\mathscr{W}''$ in that neighbourhood. Now, from elementary calculus we know that $\mathscr{W}''$ (and hence $\mathscr{F}$) must have different signs in sufficiently small punctured neighbourhoods around adjacent zeros of odd multiplicity of $\mathscr{W}'$. Consequently, $\mathscr{F}$ must change sign at some point in between adjacent zeros of odd multiplicity of $\mathscr{W}'$, and hence

$$O_Z(\mathscr{F}) \geqslant O_Z(\mathscr{W}') - 1 \geqslant O_Z(\mathscr{W}), \tag{4.22}$$

which establishes the second inequality in lemma 1.

If we use (4.9b), we can obtain the following expression for $\mathscr{H}$

$$\mathscr{H} = 2\int_{x_c}^{x} \frac{t(\Omega_1+\Omega_2)\mathscr{W}}{(1-t^2)^2}\mathrm{d}t + 2\frac{x}{(1-x)^2}\int_x^1 \frac{(\Omega_1+\Omega_2)\mathscr{W}}{(1+t)^2}\mathrm{d}t + \mathscr{T} \tag{4.23}$$

and hence we have for $\mathscr{H}'$

$$\mathscr{H}'(x) = 2\frac{1+x}{(1-x)^3}K(x), \tag{4.24}$$

$$K(x) = \int_x^1 \frac{(\Omega_1 + \Omega_2)\mathscr{W}}{(1+t)^2}\,\mathrm{d}t. \tag{4.25}$$

Since the coefficient in front of K in (4.24) is positive for all $x \in (-1, 1)$, we have

$$O_Z(K) = O_Z(\mathscr{H}'). \tag{4.26}$$

Furthermore, $K(1) = 0$ and thus by Rolle's theorem

$$O_Z(K') \geqslant O_Z(K). \tag{4.27}$$

However, we have

$$K'(x) = -\frac{(\Omega_1 + \Omega_2)\mathscr{W}}{(1+x)^2}, \tag{4.28}$$

where the coefficient in front of $\mathscr{W}$ is negative in all of $[x_c, 1)$. Consequently,

$$O_Z(\mathscr{W}) = O_Z(K') \geqslant O_Z(K) = O_Z(\mathscr{H}'), \tag{4.29}$$

which concludes the proof of the lemma. □

Proof of Proposition 1.

Assume that $O_Z(\mathscr{W}) = n$ where n is an arbitrary positive integer. We will establish that $O_Z(\mathscr{H}') \geqslant n + 1$. From the first two inequalities in lemma 1 we obtain

$$O_Z(\mathscr{G}) \geqslant O_Z(\mathscr{F}) \geqslant n \tag{4.30}$$

and in a right neighbourhood of each zero of odd multiplicity of $\mathscr{G}$, $\mathscr{G}$ and $\mathscr{G}'$ have the same sign, which is the opposite of the sign they have at the corresponding position at adjacent zeros of odd multiplicity of $\mathscr{G}$. Hence, (4.15) implies that $\mathscr{H}'$ changes sign at least once between adjacent zeros of odd multiplicity of $\mathscr{G}$. This establishes that $O_Z(\mathscr{H}') \geqslant n - 1$, and thus we must find two more zeros. To this end, let x_l and x_r denote the left- and the rightmost of the zeros of odd multiplicity of $\mathscr{G}$ in the open interval $(x_c, 1)$. Remember that the zeros found so far are located strictly between x_l and x_r. Our next task will be to prove that $\mathscr{H}'$ has at least n zeros of odd multiplicity in the semi-open interval $(x_l, 1)$.

If we do not have equality in both of the first two inequalities in lemma 1, we know that $\mathscr{G}$ has at least $n + 1$ zeros of odd multiplicity in $(x_c, 1)$, and consequently $\mathscr{H}'$ has at least n zeros of odd multiplicity in the open interval $(x_l, 1)$. Therefore, we may assume that equality holds in two first inequalities in lemma 1, i.e. that

$$O_Z(\mathscr{G}) = O_Z(\mathscr{F}) = n. \tag{4.31}$$

Let q denote the sign of $\mathscr{W}$ in a right neighbourhood of x_c. From the proof of the second inequality in lemma 1 we find that for this inequality to be an equality it is required that in a sufficiently small punctured neighbourhood of the leftmost zero of odd multiplicity of $\mathscr{W}'$ we must have that $\operatorname{sign}(\mathscr{F}) = -q$. Furthermore, the proof of the second inequality in lemma 1 also tells us that equality can only occur if $\mathscr{F}$ has constant sign to the left of the leftmost zero of odd multiplicity of $\mathscr{W}'$. This in turn implies that $\operatorname{sign}(\mathscr{F}) = -q$ in a right neighbourhood of x_c. Together with Sturm's second comparison theorem this implies that $\operatorname{sign}(\mathscr{G}) = -q$ in a right neighbourhood of x_c. Thus, (4.31) tells us that $\operatorname{sign}(\mathscr{G}(x)) = (-1)^{n+1}q$ for all $x \in (x_r, 1)$. Hence,

$$\operatorname{sign}(\mathscr{H}'(x)) = (-1)^{n+1}q, \tag{4.32}$$

for x in a right neighbourhood of x_r, since $\mathscr{G}$ and $\mathscr{G}'$ have the same sign in a sufficiently small right neighbourhood of a zero of odd multiplicity of $\mathscr{G}$.

On the other hand, successive applications of l'Hospital's rule yield

$$\lim_{x\to 1^-} \mathscr{H}'(x) = \lim_{x\to 1^-} \frac{(1-x)\mathscr{G}' + 2\mathscr{G}}{(1-x)^3} = \tfrac{1}{6}G'''(1). \tag{4.33}$$

From (2.2*b*) we find that

$$\mathscr{G}''' = -\frac{2}{1-x^2}[(\Omega_1^2)' - (\Omega_2^2)']. \tag{4.34}$$

However, we know that $\Omega_1 > 0$ and $\Omega_2 > 0$ which implies that the sign of $(\Omega_1^2) - (\Omega_2^2)$ is the same as the sign of $\mathscr{W}$. Furthermore, both $(\Omega_1^2) - (\Omega_2^2)$ and $\mathscr{W}$ are zero at 1, and hence

$$\operatorname{sign}[(\Omega_1^2)'(x) - (\Omega_2^2)'(x)] = -\operatorname{sign}(\Omega_1^2(x) - \Omega_2^2(x)) = -\operatorname{sign}(\mathscr{W}(x)), \tag{4.35}$$

in some left neighbourhood of 1. However, by definition we have

$$\operatorname{sign}(\mathscr{W}(x)) = (-1)^n q, \tag{4.36}$$

in this neighbourhood. To summarize, for x in some left neighbourhood of 1 we have

$$\begin{aligned}\operatorname{sign}(\mathscr{H}'(x)) &= \operatorname{sign}(\mathscr{G}'''(x)) = -\operatorname{sign}[(\Omega_1^2)'(x) - (\Omega_2^2)'(x)] \\ &= \operatorname{sign}(\mathscr{W}(x)) = (-1)^n q.\end{aligned} \tag{4.37}$$

A comparison between (4.32) and (4.37) tells us that $\mathscr{H}'$ changes sign somewhere between x_r and 1. Hence, we have proved that $\mathscr{H}'$ has at least n zeros of odd multiplicity in the open interval $(x_l, 1)$.

So far, our analysis has been the same for all three of our cases, but now will use different methods to establish that for each of the three cases there is a zero of odd multiplicity of $\mathscr{H}'$ in the open interval (x_c, x_l). We will establish this for the three cases, one at a time.

Case (*a*). We now know that $\mathscr{G}(x_c) = \mathscr{G}(x_l) = 0$, and that $\mathscr{G}$ does not change sign in (x_c, x_l). Hence, the real analyticity of $\mathscr{G}$ implies that $\mathscr{G}'$ has different signs in a right neighbourhood of x_c and in a left neighbourhood of x_l. Furthermore, the multiplicity of a zero of $\mathscr{G}'$ at x_c or x_l is one order lower than the corresponding multiplicity of the zeros of $\mathscr{G}$ at these points. Hence, (4.15) implies that $\mathscr{H}'$ has the same sign as $\mathscr{G}'$ in these neighbourhoods. Thus, if x belongs to a right neighbourhood of x_c, and y to a left neighbourhood of x_l, we have

$$\operatorname{sign}(\mathscr{H}'(x)) = \operatorname{sign}(\mathscr{G}'(x)) = -\operatorname{sign}(\mathscr{G}'(y)) = -\operatorname{sign}(\mathscr{H}'(y)). \tag{4.38}$$

Consequently, $\mathscr{H}'$ changes sign in (x_c, x_l).

Case (*b*). We now know that $\mathscr{G}'(x_c) = \mathscr{G}(x_l) = 0$. If in addition, $\mathscr{G}(x_c) = 0$ then we can apply the argument in case (*a*), so we may assume that $\mathscr{G}(x_c) \neq 0$. Hence, $\operatorname{sign}(\mathscr{H}'(x_c)) = \operatorname{sign}(\mathscr{G}(x_c))$. In addition, we know that the sign of $\mathscr{G}$ remains unchanged until x_l. Furthermore, in a left neighbourhood of x_l we know that the sign of $\mathscr{G}'$ is the opposite to that of $\mathscr{G}$. Since the multiplicity of a zero of $\mathscr{G}'$ at x_l is one order lower than the corresponding multiplicity of the zero of $\mathscr{G}$ at this point, it is clear from (4.15) that $\mathscr{H}'$ has the same sign as $\mathscr{G}'$ in this neighbourhood. Let, therefore, y be a point in a left neighbourhood of x_l, we then have

$$\begin{aligned}\operatorname{sign}(\mathscr{H}'(x_c)) &= \operatorname{sign}(\mathscr{G}(x_c)) = \operatorname{sign}(\mathscr{G}(y)) \\ &= -\operatorname{sign}(\mathscr{G}'(y)) = -\operatorname{sign}(\mathscr{H}'(y)).\end{aligned} \tag{4.39}$$

Consequently, $\mathscr{H}'$ must change sign in the interval (x_c, x_l).

Case (c). If $x_c = 0$, (4.12) implies that $\mathscr{G}(x_c) = 0$, which means that case (c) coincides with case (a). Let us therefore assume that $x_c > 0$, and use (4.12) to show that

$$\mathscr{H}'(x_c) = \frac{(1 + x_c)\mathscr{G}'(x_c)}{(1 - x_c)^3}. \tag{4.40}$$

If $\mathscr{G}' = 0$, we have case (b) above, and we may therefore assume that this is not the case. Hence, we have

$$\operatorname{sign}(\mathscr{H}'(x_c)) = \operatorname{sign}(\mathscr{G}'(x_c)). \tag{4.41}$$

By an argument entirely analogous to that in the previous two cases we find that for y in a left neighbourhood of x_l

$$\operatorname{sign}(\mathscr{H}'(y)) = \operatorname{sign}(\mathscr{G}'(y)). \tag{4.42}$$

Now since $x_c > 0$, (4.12) implies that

$$\operatorname{sign}(\mathscr{G}'(x_c)) = \operatorname{sign}(\mathscr{G}(x_c)). \tag{4.43}$$

Therefore, for $\mathscr{G}(x_l) = 0$ to be zero, the sign of $\mathscr{G}'$ in a left neighbourhood of x_l must be the opposite of that of $\mathscr{G}(x_c)$ and consequently to that of $\mathscr{G}'(x_c)$. This implies that for y in a left neighbourhood of x_l

$$\begin{aligned}\operatorname{sign}(\mathscr{H}'(x_c)) &= \operatorname{sign}(\mathscr{G}'(x_c)) = \operatorname{sign}(\mathscr{G}(x_c)) \\ &= \operatorname{sign}(\mathscr{G}(y)) = -\operatorname{sign}(\mathscr{G}'(y)) \\ &= -\operatorname{sign}(\mathscr{H}'(y)),\end{aligned} \tag{4.44}$$

which proves that $\mathscr{H}'$ must have a zero in (x_c, x_l). (Note that if $x_c < 0$ then the second and all subsequent inequalities above are inverted and hence we cannot establish that $\mathscr{H}'$ must have a zero of odd multiplicity in (x_c, x_l). This is the only step of the proof which fails in this case.)

We have thus established that for all our three cases $\mathscr{H}'$ has at least one zero of odd multiplicity in (x_c, x_l), in addition to the at least n such zeros we had already shown that it had in the disjoint interval $(x_l, 1)$. To summarize, we have proved that

$$O_Z(\mathscr{H}') \geqslant n + 1. \tag{4.45}$$

However, if we use the third inequality in lemma 1 this tells us that

$$O_Z(\mathscr{W}) \geqslant O_Z(\mathscr{H}') \geqslant n + 1, \tag{4.46}$$

which clearly contradicts $O_Z(\mathscr{W}) = n$ for any finite n. This concludes the proof of the proposition for $n \geqslant 1$.

The case of $n = 0$ is almost the same. Indeed, in this case we know that $\operatorname{sign}(\mathscr{W}) = q$ in the entire interval $(x_c, 1)$ except at a finite number of points where $\mathscr{W}$ have zeros of even multiplicity. By exactly the same argument as for $n > 1$, for x in some left neighbourhood of 1 we have

$$\begin{aligned}\operatorname{sign}(\mathscr{H}'(x)) &= \operatorname{sign}(\mathscr{G}'''(x)) \\ &= -\operatorname{sign}[(\Omega_1^2)'(x) - (\Omega_2^2)'(x)] \\ &= \operatorname{sign}(\mathscr{W}(x)) \\ &= (-1)^n q.\end{aligned} \tag{4.47}$$

In addition, we know that $\mathscr{W}'$ must have at least one zero, and as in the above we know that either $\operatorname{sign}(\mathscr{F}) = -q$ in a punctured neighbourhood of this point or $\mathscr{F}$ must have a zero. In the latter case, $\mathscr{G}$ must have a zero in $[x_c, 1]$ and in the former case, either $\mathscr{F}$ has constant sign throughout the interval, or $\mathscr{G}$ has a zero. If $\mathscr{G}$ has a zero, x_l, the same arguments as for $n \geqslant 1$ may be used for each of the cases (a)–(c) to assure that $\mathscr{H}'$ has a zero in (x_c, x_l). On the other hand, if $\mathscr{F}$ has constant sign, then for y in a right neighbourhood of x_c we must have $\operatorname{sign}(\mathscr{F}(y)) = -q$, and hence by Sturm's second comparison theorem it follows that $\operatorname{sign}(\mathscr{G}(y)) = -q$. In case ($b$), this immediately implies that

$$\operatorname{sign}(\mathscr{H}'(y)) = -q. \tag{4.48}$$

For cases (a) and (c), the same conclusion holds, since in both cases $\mathscr{G}$ and $\mathscr{G}'$ have the same sign in a right neighbourhood of x_c. For case (a), this is because x_c is a zero of $\mathscr{G}$ and any continuously differentiable function has the same sign as its derivative in a sufficiently small right neighbourhood of a zero. For case (c) this is a direct consequence of (4.12) when $x_c > 0$, and when $x_c = 0$ cases (c) and (a) coincide.

Hence, we have shown that either $\mathscr{G}$ and hence $\mathscr{H}'$ has a zero in $(x_c, 1)$ or both (4.47) and (4.48) must hold. From this, it is evident that $\mathscr{H}'$ must have at least one zero in $(x_c, 1)$. Now the final inequality of lemma 1 implies that $O_Z(\mathscr{W}) \geqslant 1$. This concludes the proof of the proposition. □

4.2. *Proof of theorem 2*

Proof. We may easily obtain an upper bound for C_0. To do this notice that if a bounded function U satisfies the boundary conditions then elementary calculus tells us that U'' must change sign somewhere strictly between x_c and x_s. Hence, (3.8) implies that at this point either $U = 0$ or $w = 0$ where

$$w(x) = \frac{C}{2(1+x)^2} - \frac{(x-x_c)p_1(x)}{2(1-x^2)^2}. \tag{4.49}$$

However, we have required that $U \neq 0$ and thus there must be a point strictly between x_c and x_s where $w = 0$. It is easily seen that this requires that $0 \leqslant C < (x_s - x_c)^2/[2(1-x_s^2)(1-x_c)^2]$.

Since the zeros of w are zeros of a second-degree polynomial, it is easily seen by studying the signs of w at x_c and x_s that it must have precisely one first-order zero, x_t in the domain $[x_c, x_s]$. Hence, we want to solve

$$U'' + \Gamma_c^2 w(x) U = 0, \tag{4.50}$$

in $[x_c, x_s]$ where w has a first-order zero at $x = x_t$, and is non-zero elsewhere. This problem is known as a second-order ordinary differential equation with a (first-order) turning (transition) point.

Fortunately, the asymptotic solution for this problem is well known (Abramowitz & Stegun 1970, p. 451) and is given by

$$U^a(x) = a\operatorname{Ai}(-\Gamma_c^{2/3}\xi(x)) + b\operatorname{Bi}(-\Gamma_c^{2/3}\xi(x)), \tag{4.51}$$

where Ai and Bi are the Airy functions of the first and second kind, respectively, and ξ satisfies

$$\xi(x) = \begin{cases} \left[\dfrac{3}{2}\displaystyle\int_x^{x_t} w(z)^{1/2}\mathrm{d}z\right]^{2/3} & \text{for } x_c \leqslant x \leqslant x_t. \\ -\left[\dfrac{3}{2}\displaystyle\int_{x_t}^{x} (-w(z))^{1/2}\mathrm{d}z\right]^{2/3} & \text{for } x_t \leqslant x \leqslant x_s. \end{cases} \tag{4.52}$$

Let $\xi_c = \xi(x_c)$ and $\xi_s = \xi(x_s)$. The function

$$V(\xi) = a\mathrm{Ai}(-\Gamma_c^{2/3}\xi) + b\mathrm{Bi}(-\Gamma_c^{2/3}\xi) \tag{4.53}$$

has an infinite number of positive zeros, and hence, since $\xi_c > 0$, it will have zeros for $0 \leqslant \xi \leqslant \xi_c$ when Γ_c is large enough unless we change x_t with increasing Γ_c to make sure that $-\Gamma_c^{2/3}\xi_c > \alpha_1$ holds for all values of Γ_c, where α_1 denotes the first (negative) zero of $a\mathrm{Ai}(x) + b\mathrm{Bi}(x)$. However, this condition implies that there must be a negative $\xi^\dagger$ such that $\xi_s \leqslant \xi^\dagger$ for all values of Γ_c. Consequently, the asymptotic formulae for the Airy functions (see e.g. Abramowitz & Stegun 1970) yield

$$\frac{\mathrm{Bi}'(-\Gamma_c^{2/3}\xi_s)}{\mathrm{Ai}'(-\Gamma_c^{2/3}\xi_s)} = O(\exp\,(2(-\xi^\dagger)^{3/2}\Gamma_c)). \tag{4.54}$$

Hence, the condition $U'(x_s) = 0$ implies that $b/a = O(\exp\,(-2(-\xi^\dagger)^{3/2}\Gamma_c))$. However, for negative x, $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are of the same order of magnitude, and hence the condition $U'(x_c) = 0$ is asymptotically satisfied if

$$\Gamma_c^{2/3}\xi_c = -\alpha_1', \tag{4.55}$$

where $\alpha_1' \approx -1.01879297$ is the first zero of Ai'.

To calculate ξ_c, we make a change of variables in (4.52) to obtain

$$\xi_c = \left[\tfrac{3}{2}(x_t - x_c)\int_0^1 w(x_c + (x_t - x_c)y)^{1/2}\mathrm{d}y\right]^{2/3}. \tag{4.56}$$

Suppose that $\lim_{\Gamma_c\to\infty} x_t = x^* \neq x_c$. In this case, (4.56) implies that $\xi_c \to \xi^* \neq 0$, but according to (4.55) $\xi_c = O(\Gamma_c^{-2/3})$ which is a contradiction. We have thus established that $\lim_{\Gamma_c\to\infty} x_t = x_c$. To obtain an asymptotic expression for ξ_c we now make asymptotic expansions in $\Gamma_c^{-2/3}$ of x_t and C:

$$x_t = x_c + x_{t1}\Gamma_c^{-2/3} + \cdots \tag{4.57}$$

$$C = C^0 + C_I\Gamma_c^{-2/3} + \cdots \tag{4.58}$$

where the dots denote terms of higher order. First, recall that x_t was the zero of w in $[x_c, x_s]$ which is clearly determined by C. By substituting (4.58) into $w(x) = 0$ and identifying terms we obtain the relations

$$C^0 = 0, \tag{4.59}$$

$$x_{t1} = \frac{C_I(1 - x_c)^2}{p_1(x_c)}. \tag{4.60}$$

If we use (4.60) and Taylor's formula we obtain the following asymptotic formula

$$w(x_c + (x_t - x_c)y) = \frac{C_I}{2(1 + x_c)^2}\Gamma_c^{-2/3}(1 - y) + O(\Gamma_c^{-4/3}). \tag{4.61}$$

After an additional application of Taylor's formula and integration this yields

$$\xi_c^{3/2} = \frac{C_I^{3/2}(1 - x_c)^2}{2^{1/2}p_1(x_c)(1 + x_c)}\Gamma_c^{-1}(1 + O(\Gamma_c^{-2/3})). \tag{4.62}$$

Finally, when this is compared with (4.55) we obtain the following condition for C_I

$$C_I = 2^{1/3}(-\alpha_1')\left(\frac{p_1(x_c)(1 + x_c)}{(1 - x_c)^2}\right)^{2/3} = \frac{2^{1/3}(-\alpha_1')}{(1 - x_c)^2}\left(\frac{(x_s - x_c)(1 + x_c)}{1 + x_s}\right)^{2/3}, \tag{4.63}$$

which concludes the proof of the theorem. □

5. Conclusion

The uniqueness result proved in this paper tells us that within the class of conically self-similar free-vortex solutions, a solution is uniquely determined by the opening angle of the bounding conical streamsurface, as well as the circulation and the radial velocity thereon. It is also shown that instead of the radial velocity we may take the surface radial tangential stress or the surface pressure as a parameter. These combinations of parameters can thus be used to control a conically self-similar free-vortex solution in terms of its properties at the bounding streamsurface only.

In this paper, explicit formulae have been derived in the high Γ_c-limit, which interrelate the opening angle of the separating cone in a two-cell flow and either of the surface radial tangential stress or the surface pressure for given values of the opening angle of the bounding streamsurface and the circulation thereon. One striking feature of these formulae is that they show that the value of the opening angle of the separating cone in a two-cell flow is independent of the value of the viscosity when it is low enough, i.e. we have Reynolds-number invariance for high Reynolds numbers. Some numerical checks have been performed, which show that whereas the lowest-order formulae require high values of Γ_c to reach the asymptotic regime, the refined versions of the formulae are accurate even for moderate values of Γ_c.

The uniqueness question for the problem of Yih *et al.* (1982) has been resolved with surprising results. For flows within a cone ($x_c \geqslant 0$) a uniqueness result is proved, which assures that no more than one solution can occur. For external flows ($x_c < 0$), this situation is different. Indeed, a specific case has been found numerically where at least two solutions exist. This striking property has been given a physical explanation based on a recent deciphering of the exact physical meaning of the problem itself.

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